Dehn Twists

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Modern topology uses a lot of algebraic invariants (homotopy, homology groups, etc.)

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Definition (Mapping class group)

Let S be a connected, orientable surface, possibly with punctures. Then, the *mapping class group* of S is

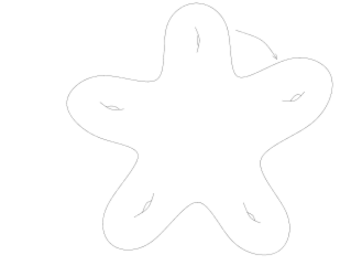
$$\begin{aligned}\mathrm{Mod}(S) &= \pi_0(\mathrm{Homeo}^+(S, \partial S)) \\ &\cong \mathrm{Homeo}^+(S, \partial S)/\text{isotopy equiv}\end{aligned}$$

In other words, these are the orientation-preserving homeomorphisms of a surface up to homotopy. Elements are called mapping classes.

Example

Let $S_{g,n}$ with genus g and n punctures.

S	Mod(S)
$D^2, S^2, S_{0,1}$	trivial
$S_{0,2}$	$\mathbb{Z}/2$
$S_{0,3}$	Σ_3
$A = S^1 \times [0,1]$	$\mathbb{Z}$
T^2	$\mathrm{SL}(2,\mathbb{Z})$
$S_{0,4}$	$\mathrm{PSL}(2,\mathbb{Z}) \ltimes (\mathbb{Z}/2 \times \mathbb{Z}/2)$



What might the generators of Mod(S) be?

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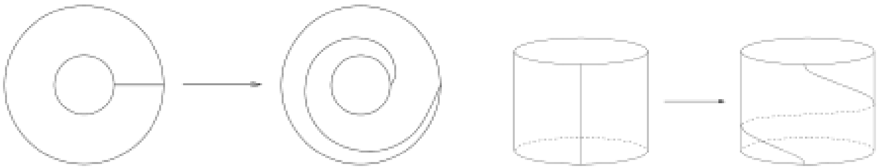

Figure: Two views of a Dehn twist

- Above, $T : A \to A$ is a map on the annulus $S^1 \times [0, 1]$ given by $T(\theta, t) = (\theta + 2\pi t, t)$.
- This map fixes the boundary ∂A.

Definition (Dehn Twist)

Let N be a regular neighborhood about a simple, closed curve a in S. Take an orientation-preserving homeomorphism $\phi : A \to N$. Then, $T_a : S \to S$ is a *Dehn twist*.

$$T_a(x) = \begin{cases} \phi \circ T \circ \phi^{-1}(x) & \text{if } x \in N \\ x & \text{if } x \in S \setminus N \end{cases}$$

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2. T_α are well-defined as an elements of Mod(S).

Example

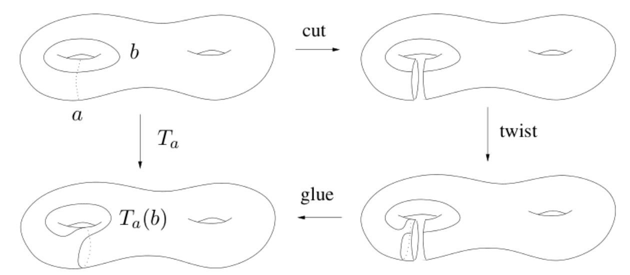

Figure: Dehn twist on torus

Question: How do Dehn twists interact in Mod(S)?

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Theorem

For two isotopy classes α, β of simple closed curves,

$$i(\alpha, \beta) = 0 \iff T_\alpha T_\beta = T_\beta T_\alpha$$

In other words, the Dehn twists commute iff the curves do not intersect.

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Definition

The geometric intersection number between classes of curves α, β is

$$i(\alpha, \beta) = \min\{|a \cap b| : a \in \alpha, b \in \beta\}$$

Example

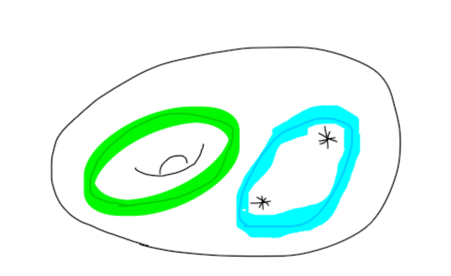

Figure: Torus with two punctured points

Idea: We get a nice result if the curves don't interact. What about higher numbers of intersections?

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Theorem (Braid Relation)

If α and β are isotopy classes of simple closed curves with $i(\alpha, \beta) = 1$, then

$$T_\alpha T_\beta T_\alpha = T_\beta T_\alpha T_\beta$$

Note equivalent to $(T_a T_b) T_a (T_a T_b)^{-1} = T_b$

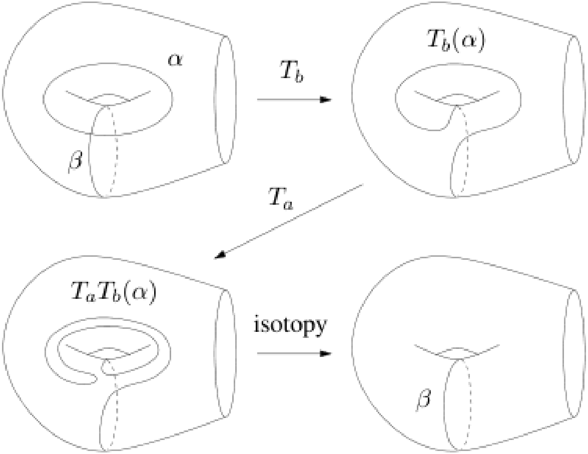

Figure: Pictorial argument of $T_a T_b(a) = b$

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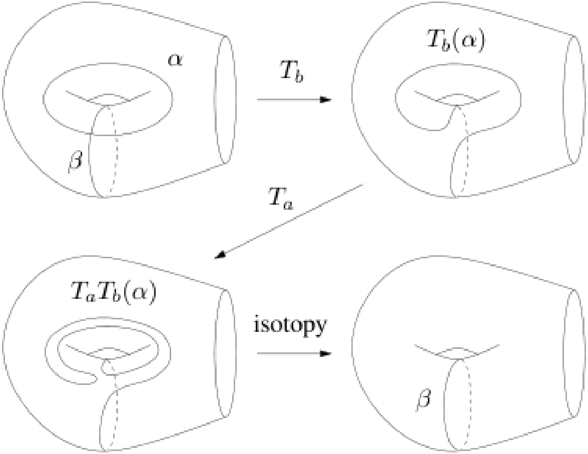

Figure: Pictorial argument of $T_a T_b(a) = b$

Surprisingly, the converse is true as well!

Again, we want to look at higher intersection numbers.

Theorem

Let α, β be isotopy classes of simple, closed curves and $i(\alpha, \beta) \geq 2$. Then, the group generated by $\{T_a, T_b\}$ is free.

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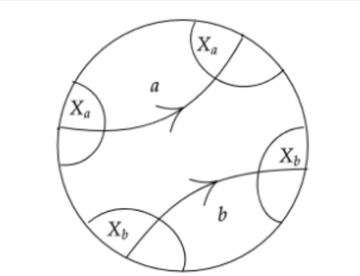
Lemma (Ping-Pong)

Let a group G act on a set X. Pick $g_1, \ldots, g_n \in G$ and disjoint, nonempty subsets $X_1, \ldots, X_n$ of X, such that for any $i \neq j$, we have

$$g_i^k(X_j) \subset X_i \text{ for all } k$$

Then, the group generated by g_i is free of rank n.

Example (Hyperbolic transformations)



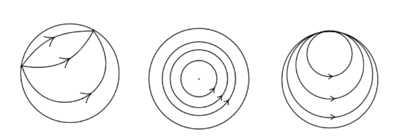


Figure: Hyperbolic isometries

With these theorems, we get a nice classification of groups generated by two Dehn twists.

	Group generated by T_a^j, T_b^k
$i(a,b)=0, a=b$	$\langle T_a^j, T_b^k\rangle \approx \langle x,y \mid x=y\rangle \approx \mathbb{Z}$
$i(a,b)=0, a\neq b$	$\langle T_a^j, T_b^k\rangle \approx \langle x,y \mid xy=yx\rangle \approx \mathbb{Z}^2$
$i(a,b)=1$	$\langle T_a, T_b\rangle \approx \langle x,y \mid xyx=yxy\rangle$ $\langle T_a^2, T_b\rangle \approx \langle x,y \mid xyxy=yxyx\rangle$ $\langle T_a^3, T_b\rangle \approx \langle x,y \mid xyxyxy=yxyxyx\rangle$ $\langle T_a^j, T_b^k\rangle \approx \langle x,y \mid \rangle \approx F_2$ otherwise
$i(a,b)\geq 2$	$\langle T_a^j, T_b^k\rangle \approx \langle x,y \mid \rangle \approx F_2$

- One can notice there are two other cases for $i(a,b)=1$.
  - $\langle T_a^2, T_b\rangle$ is index 3 subgroup of B_3 (fix "first" strand)
  - $\langle T_a^3, T_b\rangle$ is index 8 of B_3 (Luis Paris)

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For any orientable, compact surface S, the mapping class group is generated by Dehn twists.

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- Lickorish twist theorem - Possible with $3g - 1$ explicit curves!
- In fact, Humpreis (1978) showed that $2g + 1$ curves suffice and is sharp.
- Classification of groups generated by 3 Dehn twists is wide-open.

Main Reference

- ”A Primer on Mapping Class Groups” by Farb & Margalit

Acknowledgments

- Thanks to Ellis for being my wonderful mentor this semester
- And the whole Directed Reading Program, especially the organizers Maxine and Léo

Questions?